Commuting Hopf-Galois Structures on a Separable Extension

(Or: You only have to work half as hard as you think you do)

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24th of May, 2016

Hopf-Galois Module Theory

- Let L/K be a finite separable extension of local or global fields (in any characteristic).
- This extension may admit a number of Hopf-Galois structures.
- Each of these gives a different context in which to study the fractional ideals of L.
- Given a fractional ideal $\mathfrak{B}$ of L and a Hopf algebra H giving a Hopf-Galois structure on L/K, define

$$\mathfrak{A}_H = \{h \in H \mid h \cdot x \in \mathfrak{B} \text{ for all } x \in \mathfrak{B}\},$$

  and study the structure of $\mathfrak{B}$ as an $\mathfrak{A}_H$-module.
- It might be interesting to make comparisons between these different contexts:
  - For some extensions it is possible to identify the structure(s) giving the "best" description of $\mathfrak{B}$.
  - At the other extreme, for some extensions $\mathfrak{B}$ is free over its associated order in each of the Hopf-Galois structures.

Greither-Pareigis Theory

Theorem (Greither and Pareigis)

Let E/K be the Galois closure of L/K, with group G. Let $G_L = Gal(E/L)$, and let $X = G/G_L$.

- *There is a bijection between regular subgroups N of $Perm(X)$ normalized by $\lambda(G)$ and Hopf-Galois structures on L/K.*
- *The Hopf algebra giving the Hopf-Galois structure corresponding to the subgroup N is $E[N]^G$.*
- *The action of an element of such a Hopf algebra on $x \in L$ is given by*

$$\left(\sum_{n \in N} c_n n \right) \cdot x = \sum_{n \in N} c_n n^{-1}(\overline{1_G})[x].$$

- Think of a Hopf algebra produced by this theorem as coming with its action on L.

The Canonical Nonclassical Structure

- If L/K is Galois with group G then Hopf-Galois structures on L/K correspond to regular subgroups of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$.
- Two examples are $\lambda(G)$ itself and $\rho(G)$.
- The action of $\lambda(G)$ on $\rho(G)$ by conjugation is trivial, so we have:

$$L[\rho(G)]^G = L^G[\rho(G)] = K[\rho(G)],$$

  and this subgroup corresponds to the classical structure.
- If G is abelian then $\lambda(G) = \rho(G)$, but if G is nonabelian then the subgroup $\lambda(G)$ corresponds to a canonical nonclassical Hopf-Galois structure on L/K. In this case the action of $\lambda(G)$ on itself by conjugation is not trivial, so we have

$$H_\lambda := L[\lambda(G)]^G \neq K[\lambda(G)].$$

Canonical Nonclassical Module Structure

- Suppose that L/K is actually Galois, with nonabelian Galois group G.

Theorem (PT)

An element $x \in L$ generates L as an H_λ-module if and only if it generates L as a $K[G]$ module.

Theorem (PT)

A fractional ideal $\mathfrak{B}$ of L is free over its associated order in H_λ if and only if it is free over its associated order in $K[G]$.

- The proofs of these revolve around the fact that

$$\sigma(h \cdot x) = h \cdot \sigma(x) \text{ for all } \sigma \in G,\ h \in H,\ x \in L,$$

and this holds since $\lambda(G)$ and $\rho(G)$ commute inside $\mathrm{Perm}(G)$.

Centralizers of Regular Subgroups

- In fact, in the Galois case $\rho(G)$ is precisely the centralizer of $\lambda(G)$ inside $\mathrm{Perm}(G)$.
- More generally, in the case that L/K is separable with Galois closure E/K, $G = \mathrm{Gal}(E/K)$, $G_L = \mathrm{Gal}(E/L)$ and $X = G/G_L$, we have:

Lemma

If N is a regular subgroup of $Perm(X)$ normalized by $\lambda(G)$ then $N' = Cent_{Perm(X)}(N)$ is a regular subgroup of $Perm(X)$ normalized by $\lambda(G)$.

- If H denotes the Hopf algebra $E[N]^G$ corresponding to N, write H' for the Hopf algebra $E[N']^G$ corresponding to N'.

Properties of N, N' and H, H'

- $N' \cong N$,
- The Hopf algebra H' has the same type as H (but they are not necessarily isomorphic as algebras),
- $(N')' = N$,
- $(H')' = H$,
- $N = N'$ if and only if N is abelian,
- The Hopf-Galois structures given by H and H' coincide if and only if H is commutative.

Commuting Hopf-Galois Structures

Lemma

Let H give a Hopf-Galois structure on L/K. Then the actions of H and H' on L commute:

$$h' \cdot (h \cdot x) = h \cdot (h' \cdot x) \text{ for all } h \in H,\ h' \in H',\ x \in L.$$

Lemma

If H_1, H_2 give Hopf-Galois structures on L/K whose actions on L commute, then $H_2 = H_1'$.

Main Results

- Let L/K be a finite separable extension of local or global fields (in any charaacteristic) and let H give a Hopf-Galois structure on L/K.

Theorem

An element $x \in L$ generates L as an H-module if and only if it generates L as an H' module.

Theorem

A fractional ideal $\mathfrak{B}$ of L is free over its associated order $\mathfrak{A}_H$ in H if and only if it is free over its associated order $\mathfrak{A}'_H$ in H'.

Proof of second theorem

- Suppose that there exists $x \in \mathfrak{B}$ and $a_1, \ldots, a_n \in \mathfrak{A}_H$ such that $\{a_j \cdot x \mid i = 1, \ldots, n\}$ is an $\mathfrak{O}_K$-basis for $\mathfrak{B}$.
- Then x generates L as an H-module, so it generates L as an H'-module.
- Let $b_1, \ldots, b_n \in H'$ be defined by $b_i \cdot x = a_i \cdot x$ for each i.
- Since the actions of H, H' on L commute, for all i, j we have:

$$\begin{aligned} b_i \cdot (a_j \cdot x) &= a_j \cdot (b_i \cdot x) \\ &= a_j \cdot (a_i \cdot x), \end{aligned}$$

  and this lies in $\mathfrak{B}$ since $a_j \in \mathfrak{A}_H$ and the set $\{a_i \cdot x \mid i = 1, \ldots, n\}$ is an $\mathfrak{O}_K$-basis for $\mathfrak{B}$.
- Therefore $b_i \in \mathfrak{A}'_H$ for each i, so $\mathfrak{B}$ is a free $\mathfrak{A}'_H$-module.
- Since $(H')' = H$, the converse statement follows by interchanging the roles of H, H' in the argument above.

Properties (not) shared between associated orders

- Let $\mathfrak{B}$ be a fractional ideal of L, and let H give a Hopf-Galois structure on L/K.

Proposition

Suppose that the characteristic of K does not divide $[L : K]$. Then $\mathfrak{A}_H$ is a maximal order in H if and only if $\mathfrak{A}'_H$ is a maximal order in H'.

Conjecture

$\mathfrak{A}_H$ if a Hopf order in H if and only if $\mathfrak{A}'_H$ is a Hopf order in H'.

Properties (not) shared between associated orders

Counterexample

- Let $p \equiv 2 \pmod 3$ be an odd prime, let $K = \mathbb{Q}_p$, and let L be the splitting field of $x^3 - p$ over K. Then L/K is tamely ramified ($e = 3$) and Galois with group $G \cong D_3$.
- By Noether's theorem, $\mathfrak{O}_L$ is a free over $\mathfrak{O}_K[G]$, which is a Hopf order in $K[G]$.
- Therefore, $\mathfrak{O}_L$ is free over its associated order $\mathfrak{A}_\lambda$ in $H_\lambda = L[\lambda(G)]^G$. We have $\mathfrak{O}_L[\lambda(G)]^G \subseteq \mathfrak{A}_\lambda$.
- $\mathfrak{O}_L[\lambda(G)]^G$ is a Hopf order if and only if $G_0 \subseteq Z(G)$. In this example we have $|G_0| = 3$ but $Z(G) = 1$.
- If $\mathfrak{O}_L[\lambda(G)]^G \subsetneq \mathfrak{A}_\lambda$ and $\mathfrak{A}_\lambda$ is a Hopf order in H_λ then $\mathfrak{O}_L \otimes_{\mathfrak{O}_K} \mathfrak{A}_\lambda$ is a Hopf order in $L[\lambda(G)]$ properly containing $\mathfrak{O}_L[\lambda(G)]$. But this is impossible since $p \nmid |G|$.
- So $\mathfrak{A}_\lambda$ is not a Hopf order in $L[\lambda(G)]^G$.

Properties (not) shared between associated orders

Theorem

Let L/K be a finite extension of p-adic fields and H a Hopf algebra giving a Hopf-Galois structure on the extension. Let $\mathfrak{A}_H$, $\mathfrak{A}'_H$ denote the associated orders of $\mathfrak{O}_L$ in H, H' respectively. If either of these is a Hopf order then $\mathfrak{O}_L$ is a free module over each of them.

Hopf-Galois Scaffolds

- Now let L/K be a totally ramified extension of local fields with residue characteristic p and degree p^n, and let H be a Hopf algebra giving a Hopf-Galois structure on the extension.

Question

Is the existence of an H-scaffold related to the existence of an H'-scaffold?

Question

Is the previous question actually interesting?

Freeness of fractional ideals over their associated orders can be translated: is this sufficient?

Hopf-Galois Scaffolds of tolerance one

- The "alternative characterization" of scaffolds of tolerance one is most compatible with our approach. The ingredients are:
- a sequence $b_1, \ldots, b_n$ of integer *shift parameters*, each coprime to p, and a corresponding function $\mathfrak{b} : \mathbb{S}_{p^n} \to \mathbb{Z}$ defined by

$$\mathfrak{b}(s) = \sum_{i=1}^{n} s_{(n-i)} p^{n-i} b_i,$$

  where $s_{(n-i)}$ denotes the coefficient of p^{n-i} in the base-p representation of s;
- a family of elements $\Psi_1, \ldots, \Psi_n \in H$. We write $\boldsymbol{\Upsilon}^{(s)}$ for the set of monomials in the elements Ψ_i such that the exponents associated with each Ψ_i sum to $s_{(n-i)}$.

Hopf-Galois Scaffolds of tolerance one

- L/K has an H-scaffold of tolerance 1 if and only if:

1. $\Psi_i \cdot 1 = 0$ for all i;
2. There exists $x \in L$ such that

$$v_L\left(\mathbf{\Psi} \cdot x\right) = v_L(x) + \mathfrak{b}(s) \text{ for all } \mathbf{\Psi} \in \mathbf{\Upsilon}^{(s)} \text{ and } s \in \mathbb{S}_{p^n};$$

3. $v_L(\Psi_i^p \cdot y) > v_L(y) + b_i p^{n-i+1}$ for all i and all $y \in L \backslash \{0\}$.

Hopf-Galois Scaffolds of tolerance one

- Now suppose that L/K has an H-scaffold of tolerance at least 1, with shift parameters b_i.
- Does it also have an H'-scaffold of some tolerance, with some shift parameters?
- The element x in (2) generates L as an H-module, so it generates L as an H'-module.
- Define elements $\Theta_1, \ldots, \Theta_n \in H'$ by

$$\Theta_i \cdot x = \Psi_i \cdot x \text{ for each } i.$$

Proposition

We have $\Theta_i \cdot 1 = 0$ for each i.

Hopf-Galois Scaffolds of tolerance one

Proposition

We have $v_L(\boldsymbol{\Theta} \cdot x) = v_L(x) + \mathfrak{b}(s)$ for all $\boldsymbol{\Theta} \in \boldsymbol{\Upsilon}^{(s)\prime}$ and $s \in \mathbb{S}_{p^n}$.

Proof.

- For $i = 1, \ldots, n$ and any exponent $e \geq 1$ we have $\Theta_i^e \cdot x = \Psi_i^e \cdot x$.
- Using this, for $i \neq j$ and for any exponents $e_i, e_j \geq 1$ we have

$$\begin{aligned} \left(\Theta_i^{e_i}\Theta_j^{e_j}\right) \cdot x &= \Theta_i^{e_i} \cdot \left(\Psi_j^{e_j} \cdot x\right) \\ &= \Psi_j^{e_j} \cdot \left(\Theta_i^{e_i} \cdot x\right) \\ &= \left(\Psi_j^{e_j}\Psi_i^{e_i}\right) \cdot x. \end{aligned}$$

- Now let $\boldsymbol{\Theta} \in \boldsymbol{\Upsilon}^{(s)\prime}$. By repeatedly applying the result above, we find that $\boldsymbol{\Theta} \cdot x = \boldsymbol{\Psi} \cdot x$ for some $\boldsymbol{\Psi} \in \boldsymbol{\Upsilon}^{(s)}$.

□

Hopf-Galois Scaffolds of tolerance one

Proposition

If the elements Ψ_i forming the H-scaffold on L/K satisfy the following stronger form of (3):

$$\Psi_i^p = 0 \text{ for each } i$$

then so do the corresponding elements Θ_i of H'.

Theorem

If L/K has an H-scaffold of tolerance at least 1 such that the elements $\Psi_i \in H$ satisfy $\Psi_i^p = 0$ for each i, then L/K has an H'-scaffold of tolerance 1 with the same shift parameters.

Thank you for your attention.